Ref. :2010-D-621-en-2

Orig. : FR

S6ma

MATHEMATICS SYLLABUS SECONDARY 6th YEAR

Further level 3 period/week course

APPROVED BY THE JOINT TEACHING COMMITTEE ON THE 4th AND 5TH OF FEBRUARY 2010 IN BRUSSELS

Entry into force in September 2010

COMPULSORY TOPICS

1. *Foundations of Mathematics: sets, logic, mappings and groups (for guidance: 20 periods)*

Because in the past these topics have been neglected and the emphasis of teaching was put primarily on learning and the manipulation of these basic mathematical concepts, the aim of this unit is to change the emphasis, to reinforce the study of fundamental mathematical concepts. These foundations comprise of: sets, logic, mathematical reasoning, the concept of relations and applications, as well as the concept of groups and algebraic structures. It will be shown that mathematics is more than a technical skill and is above all a rigorous thought process. It is strongly recommended to study this unit at the start of the year because it may be used throughout the year and also referred to in the standard 5 period course.

In this unit, which is focussed on the basic mathematical foundations, the presence of technology will not lead to a great change to 'traditional' teaching approaches. It is left to the teacher to decide which technology they believe can add value to their teaching approach. In this unit the syllabus requires no fixed skills related to the technological tool.

Topic	Knowledge and Skills	USE OF TECHNOLOGY
The concept of sets	*Pupils must be able to and/or understand:* • appreciate and use the concept of sets: whole set, empty (null) set, subset, equality of two sets, the complement of a set, the union and intersection of sets, the relationship between the union and the intersection of sets (distributive of one over the other, De Morgan's laws), the cardinal number of a set, the set of all possible subsets and the cartesian product of sets	
Logic, vocabulary and reasoning and proof	*Pupils must be able to and/or understand:* • the basics of mathematical logic, namely the idea of proposition, formula, conjunction or disjunction of a statement or formula, implication, negation of a statement, a proposition or implication, equivalence, existential and universal quantifiers • what constitutes an axiom, a lemma, a theorem, a corollary, a condition - necessary and / or sufficient, the pigeonhole principle (Dirichlet principle)	

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	• implement the of classical mathematics methods of demonstration, in particular to demonstrate equality: o method of auxiliary hypothesis o method of separation of cases o proof by contrapositive o proof by contradiction o proof by counter example o proof by induction	
Relations and applications	*Pupils must be able to and/or understand:* • use the idea of binary relation, the graph of a relation (mapping diagram), order relations and equivalence relation, function, image, injection, surjection, bijection, inverse relation	
Groups	*Pupils must be able to and/or understand:* • define and use the concepts of closure, associativity, identity, invertibility (inverse) and commutativity (abelian) , regular element (If $a,b,c \in G$ such that $ab = ac$ then $b = c$), the student should know the following properties in a group: the uniqueness of the identity, the inverse element and the regularity of all the elements and their impact on solving equations. • during the course the students must meet the following: o infinite groups : ▪ sets of numbers ▪ some groups of geometric plane transformations (purely geometric) such as : ❖ translations ❖ enlargement with the same centre ❖ translations and point reflections ❖ translations, enlargement with the same centre ❖ translations and rotations.	

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	The study of these groups is discussed further in the option devoted to similarity. ▪ infinite set of affine functions. o finite groups: ▪ to establish the group table (also called Cayley table) closed under the group operation. ▪ know the specific properties of the groupn table of a group and form a table for: ❖ any group $(G,\times)$ with 2, 3 or 4 elements ❖ groups of symmetries of an equilateral triangle, rectangle and square ❖ groups of permutations of a finite set $(n \leq 4)$; ❖ groups of modulo *n*. • know and explain the definitions of homomorphism and isomorphism of groups to the previously studied groups	

2. *Determinants and linear algebra (for guidance: 25 periods approx.)*

This unit aims to deepen and generalise the concepts and methods introduced in previous years for the solution and the geometric interpretation of systems of linear equations in 2 or 3 unknowns.

Starting from the knowledge already acquired the introduction of the concepts of a matrix, determinant, linear combination, linear dependence and independence and the rank of a matrix will enable students to understand some of the concrete core ideas of 'Linear Algebra'. This will address not only the solution of any linear system but also enable an insight into the many practical applications of linear algebra for solving problems of every day life.

Depending on the culture of the teacher the first topic of this unit, systems of equations, may have already been met by student, however it serves as a useful introduction and prerequisite for this section.

It is compulsory that three methods of solving systems provided by linear algebra are selected and studied in detail from those listed. This list is by no means exhaustive and the teacher is free to explore additional applications eg the simplex method or various problems of optimization, to enhance the power of mathematical concepts introduced in this unit, if time permits.

The student should master the basic operations for systems of equations, matrices or determinants which are less than the order 3 without using a technological tool. The use of this tool will be a crucial support for all exercises involving a higher order than 3; the calculation of a power or the inverse of a matrix and for the calculation of determinants of order greater than 3. It will be the same for all problems involving one or more parameters.

The order in which the topics in this section are to be studied and the integration of applications listed in the last topic is at the discretion of the teacher.

Topic	Knowledge and Skills	USE OF TECHNOLOGY
Systems of equations: Gaussian Elimination	*Pupils must be able to and/or understand:* • recognise equivalent systems of equations as well as perform operations that can transform a system into an equivalent system • the concepts of systems of equations which are compatible and incompatible, determinate and indeterminate • recognise these systems by establishing relationships between the equations that make up the system or by interpreting geometrically systems of 2 or 3 equations with 2 or 3 variables	*Pupils must be able to and/or understand:* • introduce matrices in the technological tool • solve an $n \times n$ systems of equations • solve non-linear systems with 2 variables using the graphical tool technology. (one of the variables must be expressed in terms of the other in

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	• transform a given system of equations into an equivalent triangular form • study and solve systems by Gaussian elimination method (exchange of two lines. multiplication of a row by a nonzero number, adding a multiple of a row to another row) • apply Gaussian elimination for solving systems of equations, which depend on a parameter. • form from a concrete problem a system of equations, solve this system and interpret the solution obtained The solving and geometric interpretation of simple systems without a calculator will be limited to linear systems with 2 or 3 equations with 2 or 3 variables.	any equation) • check the validity of the steps performed • apply Gaussian elimination step by step • calculate directly the triangular matrix of the equation system • check the results in exercises which were performed without the tool
Basic concepts of matrices	*Pupils must be able to and/or understand:* • and apply the following concepts: o Order of an $m \times n$ matrix with elements $\left(a_{i,j}\right)_{1 \le i \le m, 1 \le j \le n}$ where m is the number of rows and n the number of columns, $m \times 1$ column matrix (vector), $1 \times n$ matrix, submatrix of a matrix, transposed matrix, zero matrix o $n \times n$ square matrices including symmetric matrices, triangular and diagonal matrices o write a matrix that satisfies certain conditions or represents a specific problem.	*Pupils must be able to and/or understand:* • perform operations on matrices of all types. • calculate the transpose of a matrix
Operations on matrices	*Pupils must be able to and/or understand:* • add two $m \times n$ matrices • multiply an $m \times n$ matrix by a scalar • understand that all matrices of type $m \times n$ with addition is a commutative group • multiply an $m \times p$ matrix by a $p \times n$ matrix and also an $m \times p$ by a $p \times m$.	*Pupils must be able to and/or understand:* • check the properties of matrix operations • find for a given matrix A, all matrices that commute with A, that is : $A \times X = X \times A$ • solve matrix equations

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	• that multiplication of matrices of type $n \times n$ is associative, not commutative and closed under multiplication. • calculate the inverse of a 2×2 square matrix without finding the determinant by using system of equations (restricted to simple cases) • understand that in general for a set of $n \times n$ matrices, with the operation of multiplication does not form a group. • solve simple matrix equations of the type $A \times X = B$ • write a system of linear equations in matrix form • calculate the powers of square matrices A^n for simple cases	• check the results in exercises performed without the tool
Determinants of order *n*	*Pupils must be able to and/or understand:* • calculate the determinants of 2×2 and 3×3 square matrices i.e. by using the products of diagonals (rule Sarrus), and understand that this is an algorithm that results from the solution of linear systems of 2 equations in 2 variables and 3 equations in 3 variables • understand that the determinant of an $n \times n$ matrix, where $n \geq 3$, can be calculated by considering each element $a_{i,j}$. The row *i* with a column *j* are crossed out leaving an $(n-1)\times(n-1)$ submatrix $M_{i,j}$ of the element $a_{i,j}$. The determinant of $M_{i,j}$ is the minor $m_{i,j}$. The cofactor of $a_{i,j}$ is given by $A_{i,j} = (-1)^{i+j}\, m_{i,j}$. • The determinant of the matrix is equal to the sum of the products obtained by multiplying the elements of any row (column) by their respective cofactors. • calculate the determinant of order 4 without using a technological tool. • apply to the following properties to the calculation of a determinant of order n (for simplicity the following apply to order 3 or below): o $\det(C_1, C_2, C_3) = -\det(C_2, C_1, C_3)$ o $\det(rC_1, C_2, C_3) = r\det(C_1, C_2, C_3)$	*Pupils must be able to and/or understand:* • calculate the determinant of a square matrix • check the properties of determinants • transformations of zeros in a row (or column) to find a determinant • solve problems using determinants • calculate the rank of a matrix

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	o $\det\left(C_1^{'}+C_1^{''},C_2,C_3\right)=\det\left(C_1^{'},C_2,C_3\right)+\det\left(C_1^{''},C_2,C_3\right)$ o every determinant containing a column (or row) consisting entirely of zeros is zero o a determinant with two identical columns (or two rows) is zero o a determinant with two columns (or two rows) which are a multiple of each other is zero o $\det\left(C_1+rC_2+sC_3,C_2,C_3\right)=\det\left(C_1,C_2,C_3\right)$ o $\det\left(M\times N\right)=\det\left(M\right)\cdot\det\left(N\right)$ where *M* and *N are* two square matrices • calculate the determinants containing one or more parameters and find the values of the parameters for which the determinant is zero	
Linear combinations, linear dependence and independence, rank of a matrix	*Pupils must be able to and/or understand:* • based on knowledge acquired in previous years, during the study of the set of vectors of the plane: o give the definition of a linear combination of matrices (vectors) rows or columns o define the concept of linear dependence and independence of rows or columns matrices (vectors) (use the fundamental property of linear independence: $\alpha_1C_1+\alpha_2C_2+\cdots+\alpha_nC_n=O\Rightarrow\alpha_1=\alpha_2=\cdots=\alpha_n=0$) o check linear dependence or independence of rows or columns matrices (vectors) by simple observation, by theoretical arguments or by using the fundamental property o define the concept of a nonsingular (regular) square matrix (i.e. non-zero determinant) o define the rank of an $m\times n$ or $n\times m$ matrix *A* (m$\geq$n) as at most n (the highest order of a nonsingular submatrix) or as the minimum number of linearly independent rows or columns. o determine the rank of a matrix by simple observation of its	

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	elements in simple or obvious cases, by using the determinants of submatrices, or by using the Gauss pivot method o consider the rank of a matrix depending on a parameter o use the determinants to verify the linear dependence or independence of matrix rows or columns	
Inverse of a nonsingular (regular) square matrix	*Pupils must be able to and/or understand:* • check whether a given matrix is non-singular or singular • determine the minor $m_{i,j}$, from the submatrix obtained by crossing out the ith row and jth column of the elements $a_{i,j}$ • find the cofactors A_{ij}, $A_{i,j} = (-1)^{i+j} m_{i,j}$,of the elements $a_{i,j}$ • determine the inverse of a given matrix by multiply the transposed matrix of cofactors by the inverse of the determinant of the matrix. • justify that all non-singular square matrices of order 2 form a non-commutative group with multiplication • solve systems of linear equations by writing them in matrix form and applying the matrix inverse.	*Pupils must be able to and/or understand:* • whether a matrix is nonsingular • find the rank of a matrix • calculate the inverse of a matrix • use a technological tool for problems involving parameters
Solving $m \times n$ **systems using determinants**	*Pupils must be able to and/or understand:* • know the general theorem to solve systems of $m \times n$ linear equations (Rouché-Fröbenius theorem) • applying Cramer's rule to solve systems of $m \times n$ linear equations • apply Rouché-Fröbenius theorem and Cramer's rule to solve systems of systems of linear equations of one or more parameters • solve homogeneous systems of equations All exercises without a technological tool will be limited to cases of simple systems of 2 or 3 equations with to 2 or 3 variables.	*Pupils must be able to and/or understand:* • solve $n \times n$ systems of linear equations by using Cramer's rule

Topic	Knowledge and Skills	USE OF TECHNOLOGY
Applications	*Pupils must be able to and/or understand:* • work with three of the following applications using the technological tool. The selection of the three applications is made by the teacher: o solve probability problems which stabilise in long-term (see example 1) o considering situations related to the dynamics of groups to obtain conclusions about leadership, isolated individuals, lobbyists, etc. (see example 2 and 4) o use the powers of matrices (A^n) to solve problems of graphs (see example 3) o use the product of matrices to solve problems of public health. (see Example 5) o use the product of a matrix by a vector to study the evolution of a population or migration (see examples 6 and 7) o use the product of a matrix by a vector to study migration. (see example 7) o use the product of matrices to analyse production processes (see example 8) o consider a Markov chain in one of the following or any other situation (see example 9) o planning and forecasting of population migration. (see example 10) o use the inverse of a matrix to solve problems in cryptography (see example 11) o express plane transformation by using their associated matrices (see example 12) o consider simple cases of problem by eigenvalues (see Example 13) NB: The examples will be available to the teacher in an accompanying document.	*Pupils must be able to and/or understand:* • optimise the different possibilities of using the technological tool to solve the chosen three practical applications

3. *Numerical Analysis (for guidance: 20 periods)*

This section has multiple objectives.

First, we show that the concept of an algorithm is inherent in many situations already encountered by the student; where reference has been made to the concept without actually using the word. Therefore we could say that examples, such as those listed below, are not entirely new.

On the one hand, it is advantageous to introduce numerical methods which are applied to two types of mathematical problems: searching for the zeros of a function and approximating integrals, while waiting for the latter to be developed in the 5 period course in year 7. Similarly , the implementation of these methods will require an initiation of the student in the basic principles of algorithms (management of the input and output or even assigning a value and putting it in the form of a calculation).

On the other hand, we will exploit the benefits offered by using a spreadsheet in a mathematical context.
Finally, the mathematical rigor will manifest itself in the quantifying of error. Also showing that despite the incomplete formal approach the numerical calculation must be "under control".

In summary, this unit will allow a student to improve their level of accuracy, by motivating them to consider their findings and test result.

NB: This "applied" mathematics section does not seek to replace a computer course in any way.

Topic	Knowledge and Skills	USE OF TECHNOLOGY

The concept of an algorithm	*Pupils must be able to and/or understand:*
	
	• define the concept of algorithm
	• read and understand an algorithm
	• write an algorithm for simple cases such as:
	    o Euclidean division or decimal division
	    o a square root determined by trial or error
	    o the approximation of a square root algorithm using the Babylonian algorithm
	    o the approximation of π by any method
	    o solve a quadratic equation or a system of equations
	    o find the GCD (greatest common divisor) of two numbers, especially by using the Euclidean algorithm	

Topic	Knowledge and Skills	USE OF TECHNOLOGY
	o calculate the power n, where n is a positive integer or factorial of a number (recursive algorithm)	
Solving nonlinear equations	*Pupils must be able to and/or understand:* • explain and represent methods: o search for a solution by using intervals of 10. o bisection o type $x_{n+1} = f(x_n)$ ▪ secant method, also called the Lagrange method ▪ Newton's method, also called the tangent method ▪ method for determining the fixed point (root) $f(x) = x$ • identify any assumptions or possible conditions that must be met	*Pupils must be able to and/or understand:* • application of each of the methods listed, both algebraically and using a spreadsheet • apply the algorithms corresponding to these methods, step by step • graphically plot some loops of the algorithm • assess the appropriateness (convergence or not) of an algorithm • assessment the speed of convergence of an algorithm
Calculation of areas	*Pupils must be able to and/or understand:* • explain and apply the methods: o rectangle method (midpoint rule) (for: Top-left corner, Midpoint and Top-right corner approximation) o the trapezium rule o Simpson's rule • measure the error for the first two methods, for the third method discussion of the error is left to the discretion of the teacher	*Pupils must be able to and/or understand:* • application of each of the methods listed, both algebraically and using a spreadsheet • apply the algorithms corresponding to these methods, step by step • graphically plot some loops of the algorithm • assess the speed of convergence of an algorithm

OPTIONAL MATERIAL (choose only one option)

Option A: Vector Spaces ((for guidance: 20 periods)

The vector space structure on a field is essential. It occurs naturally in geometry when we list properties of all vectors of the plane, or space, equipped with the usual operations, but also in all polynomials in one variable or all $m \times n$ matrices˜ Other examples of varying complexity could be presented. Inextricably linked with this concept is the idea of a vector subspace , which will also be considered. Hence the basic concepts of vector spaces and subspaces are defined and applied. Finally, discussing the isomorphism (one-to-one correspondence) between any vector space of dimension n and $\mathbb{R}^n$ is unavoidable.

Topic	Knowledge and Skills	USE OF TECHNOLOGY
Vector Spaces	*Pupils must be able to and/or understand:* • define a vector space (the definition of the algebraic structure of the field will be briefly described and illustrated with simple examples familiar to students ($\mathbb{Q}$,$\mathbb{R}$ or $\mathbb{C}$ or usual operations), as well as being able to distinguish between addition of vectors and by a scalar multiplication. • know the following properties: o $0 \cdot \vec{a} = \vec{0}$ o $\lambda \cdot \vec{0} = \vec{0}$ o $(-1) \cdot \vec{u} = -\vec{u}$ o $\lambda \cdot \vec{u} = 0 \Leftrightarrow \lambda = 0$ or $\vec{u} = \vec{0}$ • calculate linear combinations of vectors • know at least the examples from the current 4 th and/or 5 th year o vector space of vectors of a plane (2D) or space (3D) o vector space of polynomials o (continuous functions $(+, \mathbb{R}, \times)$) o $(\mathbb{R}^n, +, \mathbb{R}, \times)$	

Topic	Knowledge and Skills	USE OF TECHNOLOGY

Vector Subspaces	*Pupils must be able to and/or understand:*
	
	• define a subspace
	• specify sufficient conditions for a set to be a subspace
	• define a generating independent and dependent set
	• define a base, ordered basis
	• define the dimension the base
	• about the relationships (implications) between generating set independent set, basis and dimension
	• describe the existence of an isomorphism between a vector space of dimension n and $\mathbb{R}^n$
	• know at least the following examples:
	  o 1-dimensional vector space: a straight line
	  o 2-dimensional vector space: a plane
	  o 3-dimensional vector space: space (3 dimensions) and all polynomials of degree less than or equal to 2
	  o 4-dimensional vector space: all polynomials of degree less than or equal to 3 and all 2×2 square matrices
		Pupils must be able to and/or understand:
		
		• optimise the different possibilities of using the technological tool for work in a space of dimension greater than or equal to 3

Option B: Direct Isometries and direct similarities in the complex plane (for guidance: 20 periods)

This option complements the topic of complex numbers studied in the 5 period course. We will focus mainly on geometrical aspects but may occasionally use the possibilities of the methods of complex numbers to explain certain trigonometric formulas. The latter aspect can be utilised in the compulsory part of the year 7 advanced course concerned with equations and trigonometric functions. In addition, the content of this option can greatly simplify some proofs in the chapter on mathematical foundations (group structure).

opic	Knowledge and Skills	USE OF TECHNOLOGY
Geometrical interpretation of operations on complex numbers	*Pupils must be able to and/or understand:* • interpret geometrically operations on complex numbers: o the sum o product with a real number o product with a purely imaginary scalar o product with any complex number o the inverse of a complex number o the complex conjugate	
Functions ***f (z) = az + b***	*Pupils must be able to and/or understand:* • the concept of complex-valued function with a complex variable • the geometric interpretation of each of the following functions from $\mathbb{C}$ to $\mathbb{C}$ and their compositions: $z \mapsto z + a \ (a \in \mathbb{R})$ $z \mapsto kz \ (k \in \mathbb{R})$ $z \mapsto (\cos\theta + i\sin\theta) z \quad (\theta \in \mathbb{R})$ and vice versa • for functions *f* defined by the form $f(z) = az + b \ (a \in \mathbb{C}, a \neq 0)$ *:*	*Pupils must be able to and/or understand:* • perform all calculations required, including solving equations, using algebraic notation, trigonometric, exponential or matrix

opic	Knowledge and Skills	USE OF TECHNOLOGY
	o find the point(s) which are mapped to themselves o show the conservation of angles and ratios of length o give a geometric interpretation o give the inverse • form a matrix for such a function and state its inverse	
Direct Similarity	*Pupils must be able to and/or understand:* • that to each similarity a complex function is associated and vice versa • composition of similarities • groups of direct similarities (isomorphism) • the image of a line, a circle, a polygon	*Pupils must be able to and/or understand:* • construct images of points and figures that arise from transformations that have been mentioned in this option (point symmetry, reflection, translation, rotation, enlargement (reduction) and a composition of them
Direct Isometries	*Pupils must be able to and/or understand:* • associate with each direct isometry its associated complex function and vice versa • composition of isometries • groups of direct isometries (isomorphism) • the image of a line, a circle, a polygon	*Pupils must be able to and/or understand:* • construct images of points and figures that arise from transformations that have been mentioned in this option (point symmetry, reflection, translation, rotation and a composition of them)